

RICKART MODULES RELATIVE TO GOLDIE TORSION THEORY

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ABSTRACT. Let R be an arbitrary ring with identity and M a right R -module with $S = \text{End}_R(M)$. Let $Z_2(M)$ be the second singular submodule of M . In this paper, we define Goldie Rickart modules by utilizing the endomorphisms of a module. The module M is called Goldie Rickart if for any $f \in S$, $f^{-1}(Z_2(M))$ is a direct summand of M . We provide several characterizations of Goldie Rickart modules and study their properties. Also we present that semisimple rings and right Σ - t -extending rings admit some characterizations in terms of Goldie Rickart modules.

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1. INTRODUCTION

Throughout this paper R denotes a ring with identity, modules are unital right R -modules. Let M be an R -module with $S = \text{End}_R(M)$. The singular submodule of M is $Z(M) = \{m \in M \mid mI = 0 \text{ for some essential right ideal } I \text{ of } R\}$. If $M = Z(M)$, then M is called *singular* and M is *nonsingular* provided $Z(M) = 0$. The second singular submodule, in other words, the Goldie torsion submodule $Z_2(M)$ of M is defined by $Z(M/Z(M)) = Z_2(M)/Z(M)$. The module M is called Z_2 -torsion (or *Goldie torsion*) if $M = Z_2(M)$. It is evident that every singular module is Z_2 -torsion. If for any $f \in S$, $r_M(f)$ is a direct summand of M , then M is said to be a *Rickart module*. Rickart modules are introduced and investigated by Lee, Rizvi and Roman in [6]. Also, right (left) Rickart rings (or principally projective rings) initially appeared in Maeda [8], and were further studied by Hattori [4], that is, a ring is called *right (left) Rickart* if every principal right (left) ideal is projective, equivalently, the right annihilator of any single element is generated by an idempotent as a right ideal. The concept of right (left) Rickart rings has been comprehensively studied in the literature. In [2], Asgari and Haghany defined *t-Baer modules*, that is, a module M is called *t-Baer* if $t_M(I)$ is a direct summand of M for every left ideal I of S where $t_M(I) = \{m \in M \mid Im \leq Z_2(M)\}$, and they study properties of t-Baer modules. Note that for any $f \in S$, $f^{-1}(Z_2(M)) = t_M(Sf)$ and

$\bigcap_{f \in S} f^{-1}(Z_2(M)) = Z_2(M)$. Clearly, the kernel $r_M(f)$ of $f \in S$ is a submodule of $f^{-1}(Z_2(M))$.

In what follows, by \mathbb{Z} , \mathbb{Q} and \mathbb{Z}_n we denote, respectively, integers, rational numbers and the ring of integers modulo n . For a module M , $E(M)$ is the injective hull of M and $S = \text{End}_R(M)$ is the ring of endomorphisms of M .

2. GOLDIE RICKART MODULES

In this section we give our main definition, namely Goldie Rickart modules, and investigate some properties of this class of modules.

Definition 2.1. *A module M is called Goldie Rickart if $f^{-1}(Z_2(M))$ is a direct summand of M for every $f \in S$.*

It is clear that every semisimple module, every singular module and every Z_2 -torsion module is Goldie Rickart. If R is a ring with $Z_2(R_R) = R$, then every R -module M is Goldie Rickart due to $Z_2(M) = M$. For a nonsingular module M , since $r_M(f) = f^{-1}(Z_2(M))$, M is Rickart if and only if it is Goldie Rickart, however these two notions are not equivalent for any arbitrary module that will be shown later.

Proposition 2.2. *Every indecomposable Goldie Rickart module is a Rickart or Z_2 -torsion module.*

Proof. Let M be an indecomposable Goldie Rickart module and 1 denote the identity endomorphism of M . Since $1^{-1}(Z_2(M)) = Z_2(M)$, $Z_2(M) = M$ or $Z_2(M) = 0$. This implies that M is Z_2 -torsion or it is nonsingular and so it is Rickart. \square

Proposition 2.3. *Every indecomposable extending module is a nonsingular or Goldie Rickart module.*

Proof. Let M be an indecomposable extending module. $Z_2(M)$ is a direct summand of M since it is a closed submodule of M . Hence $Z_2(M) = 0$ or $Z_2(M) = M$. This implies that M is nonsingular or Goldie Rickart due to $f^{-1}(Z_2(M)) = M$ for every $f \in S$. \square

Recall that a module M has the (strong) summand intersection property if the intersection of (any) two direct summands is a direct summand of M .

Proposition 2.4. *Every t -Baer module M is Goldie Rickart. The converse holds if M has the strong summand intersection property for direct summands which contain $Z_2(M)$.*

Proof. Let M be a t -Baer module and $f \in S$. Since $t_M(Sf) = f^{-1}(Z_2(M))$, M is a Goldie Rickart module. The converse is true due to [2, Theorem 3.2]. \square

In [2] it is said that a submodule N of a module M is *t-essential* if for every submodule L of M , $N \cap L \leq Z_2(M)$ implies that $L \leq Z_2(M)$, and N is called *t-closed* if N has no t -essential extension in M . The module M is called *t-extending* if every t -closed submodule of M is a direct summand of M , while a ring R is called *right Σ -t-extending* if every free R -module is t -extending.

Proposition 2.5. *Every t -extending module is Goldie Rickart.*

Proof. Let M be a t -extending module and $f \in S$. By [2, Corollary 2.7], for any module M , $f^{-1}(Z_2(M))$ is t -closed in M . By hypothesis $f^{-1}(Z_2(M))$ is a direct summand of M . \square

By combining Proposition 2.5 with [2, Theorem 3.12] we have Theorem 2.6.

Theorem 2.6. *The following are equivalent for a ring R .*

- (1) *R is right Σ -t-extending.*
- (2) *Every R -module is t -extending.*
- (3) *Every R -module is t -Baer.*
- (4) *Every R -module is Goldie Rickart.*

We obtain the next result as an immediate consequence of [3, Theorem 2.15], [2, Theorem 3.12] and Theorem 2.6.

Proposition 2.7. *If a ring R is Morita-equivalent to a finite direct product of full lower triangular matrix rings over division rings, then every R -module is Goldie Rickart.*

We now give a useful characterization of Goldie Rickart modules by using Goldie torsion submodules.

Theorem 2.8. *A module M is Goldie Rickart if and only if $M = Z_2(M) \oplus N$ where N is a (nonsingular) Rickart module.*

Proof. Let M be a Goldie Rickart module and 1_M denote the identity endomorphism of M . Then $1_M^{-1}(Z_2(M)) = Z_2(M)$ is a direct summand of M . Let $M = Z_2(M) \oplus N$ for some submodule N of M and $f \in \text{End}_R(N)$. Hence $1_{Z_2(M)} \oplus f \in S$, say $g = 1_{Z_2(M)} \oplus f$. This implies that $g^{-1}(Z_2(M)) = Z_2(M) \oplus r_N(f)$. By assumption, $g^{-1}(Z_2(M))$ is a direct summand of M . It follows that $r_N(f)$ is a direct summand of N . Therefore N is Rickart. Since $M/Z_2(M)$ is nonsingular, N is

a nonsingular module. For the converse, assume that $M = Z_2(M) \oplus N$ where N is a (nonsingular) Rickart module. Let $f \in S$ and π_N denote the projection on N along $Z_2(M)$. Then $\pi_N f|_N \in \text{End}_R(N)$ and it can be easily shown that $f^{-1}(Z_2(M)) = Z_2(M) \oplus r_N(\pi_N f|_N)$. Since N is Rickart, $r_N(\pi_N f|_N)$ is a direct summand of N , and so $f^{-1}(Z_2(M))$ is a direct summand of M . This completes the proof. \square

In [9], if I is an ideal of a ring R , it is said that *idempotents lift strongly modulo I* if whenever $a^2 - a \in I$, there exists $e^2 = e \in aR$ (equivalently $e^2 = e \in Ra$) such that $e - a \in I$. Also a ring R is called $Z_2(R_R)$ -semiperfect [9] if $R/Z_2(R_R)$ is semisimple and idempotents lift strongly modulo $Z_2(R_R)$.

Corollary 2.9. *Let R be a $Z_2(R_R)$ -semiperfect ring. Then every R -module is Goldie Rickart.*

Proof. Let M be an R -module. Then $M = Z_2(M) \oplus N$ where N is semisimple by [9, Theorem 49]. Hence M is Goldie Rickart in virtue of Theorem 2.8. \square

In the light of [10, Theorem 2.5], if R is a QF-ring, then it is $Z_2(R_R)$ -semiperfect. Then we have the next result due to Corollary 2.9.

Corollary 2.10. *Every module over a QF-ring is Goldie Rickart.*

Due to Theorem 2.8, if M is a Goldie Rickart module, then $M/Z_2(M)$ is a Rickart module, and it is Goldie Rickart since it is nonsingular. But the converse does not hold in general, as the following example shows.

Example 2.11. Let $\mathcal{P} = \{p \in \mathbb{Z} \mid p \text{ is prime}\}$ and consider the \mathbb{Z} -module $M = \prod_{p \in \mathcal{P}} \mathbb{Z}_p$. Then $Z(M) = \bigoplus_{p \in \mathcal{P}} \mathbb{Z}_p$ and $Z_2(M) = Z(M)$. Since $M/Z_2(M)$ is extending and nonsingular, it is Rickart. But $Z_2(M)$ is not a direct summand of M , and so M is not Goldie Rickart, by Theorem 2.8.

Remark 2.12. Example 2.11 also reveals the fact that if M/N is a Goldie Rickart module for any submodule N of a module M , then M need not be Goldie Rickart. Because in Example 2.11, the module $M/Z_2(M)$ is Rickart and nonsingular, hence it is Goldie Rickart while M is not Goldie Rickart.

According to next examples, Rickart modules and Goldie Rickart modules do not imply each other.

Examples 2.13. (1) Consider the module M in the Example 2.11. It is known from there M is not Goldie Rickart. On the other hand, the endomorphism ring

S of M is $\prod_{p \in \mathcal{P}} \mathbb{Z}_p$. Since S is a von Neumann regular ring, M is Rickart by [11, Corollary 3.2].

(2) Consider \mathbb{Z}_4 as a \mathbb{Z} -module. Then \mathbb{Z}_4 is a Goldie Rickart module due to $Z(\mathbb{Z}_4) = \mathbb{Z}_4 = Z_2(\mathbb{Z}_4)$. On the other hand, for $f \in \text{End}_{\mathbb{Z}}(\mathbb{Z}_4)$ with $f(\bar{1}) = \bar{2}$, $r_{\mathbb{Z}_4}(f) = 2\mathbb{Z}_4$ is not a direct summand of \mathbb{Z}_4 . Hence \mathbb{Z}_4 is not Rickart.

Now we give a relation between the Rickart and Goldie Rickart modules.

Theorem 2.14. *Let M be a module. Then the following are equivalent.*

- (1) M is Goldie Rickart and $r_M(f)$ is a direct summand of $f^{-1}(Z_2(M))$ for any $f \in S$.
- (2) M is Rickart and $Z_2(M)$ is a direct summand of M .

Proof. (1) \Rightarrow (2) Let M be a Goldie Rickart module and $f \in S$. Then $f^{-1}(Z_2(M))$ is a direct summand of M and by hypothesis, $r_M(f)$ is a direct summand of $f^{-1}(Z_2(M))$. It follows that M is Rickart. In addition, by Theorem 2.8, $Z_2(M)$ is a direct summand of M .

(2) \Rightarrow (1) Let M be a Rickart module and $M = Z_2(M) \oplus N$ for some submodule N of M . Then N is Rickart and so M is Goldie Rickart by Theorem 2.8. The rest is clear since M is Rickart and $r_M(f)$ is a submodule of $f^{-1}(Z_2(M))$ for any $f \in S$. \square

Proposition 2.15. *The following hold for a module M .*

- (1) If M is Rickart with $Z(M)$ a direct summand of M , then M is Goldie Rickart.
- (2) If M is Goldie Rickart and R is right nonsingular, then $Z(M)$ is a direct summand of M .

Proof. (1) Let $Z(M)$ be a direct summand of M . Since $Z(M)$ is essential in $Z_2(M)$, we have $Z(M) = Z_2(M)$. Then Theorem 2.14 completes the proof.

(2) The right nonsingularity of R implies that $Z(M) = Z_2(M)$. The rest is clear because M is Goldie Rickart. \square

Lemma 2.16. *Let M be a module. Then the following are equivalent.*

- (1) M is a Goldie Rickart module.
- (2) The exact sequence $0 \rightarrow f^{-1}(Z_2(M)) \rightarrow M \rightarrow M/f^{-1}(Z_2(M)) \rightarrow 0$ is split for any $f \in S$.

Proof. Let $f \in S$ and consider the exact sequence $0 \rightarrow f^{-1}(Z_2(M)) \rightarrow M \rightarrow M/f^{-1}(Z_2(M)) \rightarrow 0$. Then M is Goldie Rickart if and only if $f^{-1}(Z_2(M))$ is a direct summand of M if and only if the exact sequence is split. \square

In the next result we give a characterization of semisimple rings by using the notion of Goldie Rickart modules.

Theorem 2.17. *The following are equivalent for a ring R .*

- (1) *Every R -module is Goldie Rickart and its Goldie torsion submodule is projective.*
- (2) *R is semisimple.*

Proof. (1) \Rightarrow (2) Let M be an R -module. Since M is Goldie Rickart, by Theorem 2.8, $M = Z_2(M) \oplus N$ for some nonsingular submodule N of M . By hypothesis $Z_2(M)$ is projective. Also by Theorem 2.6 and [2, Theorem 3.12], N is projective. Hence M is projective and so R is semisimple due to [1, Corollary 17.4].

(2) \Rightarrow (1) Let M be an R -module and $f \in S$. Consider the exact sequence $0 \rightarrow f^{-1}(Z_2(M)) \rightarrow M \rightarrow M/f^{-1}(Z_2(M)) \rightarrow 0$. By [1, Proposition 13.9], this exact sequence is split, and so M is Goldie Rickart due to Lemma 2.16. The rest is clear from [1, Corollary 17.4]. \square

Recall that a module M is called *duo* if every submodule of M is fully invariant, i.e., for a submodule N of M , $f(N) \leq N$ for each $f \in S$. Fully invariant submodules of Goldie Rickart modules are also Goldie Rickart under some conditions.

Lemma 2.18. *Let M be a Goldie Rickart module and N a fully invariant submodule of M . If every endomorphism of N can be extended to an endomorphism of M , then N is Goldie Rickart.*

Proof. Let $f \in \text{End}_R(N)$. By hypothesis, there exists $g \in S$ such that $g|_N = f$ and being M Goldie Rickart, there exists $e = e^2 \in S$ such that $g^{-1}(Z_2(M)) = eM$. Since N is fully invariant, $e|_N$ is an idempotent of $\text{End}_R(N)$. We claim that $f^{-1}(Z_2(N)) = e|_N N$. Clearly, $f^{-1}(Z_2(N)) \subseteq e|_N N$. In order to see other inclusion, let $n \in N$. Then $fe|_N n = gen \in Z_2(M) \cap N = Z_2(N)$, and so $e|_N n \in f^{-1}(Z_2(N))$. Hence we have $e|_N N \subseteq f^{-1}(Z_2(N))$. This implies that N is Goldie Rickart. \square

Recall that a module M is called *quasi-injective* if it is M -injective. It is well known that every quasi-injective module is a fully invariant submodule of its injective hull. By considering this fact, we can say the next result as an immediate consequence of Lemma 2.18.

Proposition 2.19. *Let M be a quasi-injective module. If $E(M)$ is Goldie Rickart, then so is M .*

Theorem 2.20. *Let M be a quasi-injective duo module. If M is Goldie Rickart, then so is every submodule of M .*

Proof. Let M be a Goldie Rickart module, N a submodule of M and $f \in \text{End}_R(N)$. By quasi-injectivity of M , f extends to an endomorphism g of M . Then $g^{-1}(Z_2(M)) = eM$ for some $e = e^2 \in S$. Since N is fully invariant in M , the proof follows from Lemma 2.18. \square

Proposition 2.21. *Every direct summand of a Goldie Rickart module is also Goldie Rickart.*

Proof. Let M be a Goldie Rickart module and N a direct summand of M . There exists a submodule K of M with $M = N \oplus K$. Let $f \in \text{End}_R(N)$. Hence $f \oplus 1_K \in S$, say $g = f \oplus 1_K$. Since M is Goldie Rickart, $g^{-1}(Z_2(M))$ is a direct summand of M . On the other hand, $g^{-1}(Z_2(M)) = f^{-1}(Z_2(N)) \oplus Z_2(K)$. Thus $f^{-1}(Z_2(N))$ is a direct summand of N . Therefore N is Goldie Rickart. \square

In comparison with Proposition 2.21, in general, a direct sum of Goldie Rickart modules may not be Goldie Rickart as shown below.

Example 2.22. Let R denote the ring of 2×2 upper triangular matrices over $\mathbb{Q}[x]$, M right R -module R and e_{ij} , ($1 \leq i, j \leq 2$), 2×2 matrix units. Consider the submodules $N = e_{11}R$ and $K = e_{22}R$ of M . Then $M = N \oplus K$. Note that M is a nonsingular module and $S \cong R$. It is evident that N and K are Goldie Rickart. On the other hand, for $f = e_{11}2x + e_{12}x \in S$, $r_M(f) = (-e_{12}x + 2e_{22}x)R$ is not a direct summand of M . Hence M is not Rickart, therefore it is not Goldie Rickart.

Now we investigate some conditions about when direct sums of Goldie Rickart modules are also Goldie Rickart, but more details are in the last section.

Proposition 2.23. *Let $\{M_i\}_{i \in \mathcal{I}}$ be a class of R -modules for an arbitrary index set \mathcal{I} . If $\text{Hom}_R(M_i, M_j) = 0$ for every $i, j \in \mathcal{I}$ with $i \neq j$ (i.e., for every $i \in \mathcal{I}$, M_i is a fully invariant submodule of $\bigoplus_{i \in \mathcal{I}} M_i$), then $\bigoplus_{i \in \mathcal{I}} M_i$ is Goldie Rickart if and only if M_i is Goldie Rickart for every $i \in \mathcal{I}$.*

Proof. The necessity is clear by Proposition 2.21. Conversely, let $M = \bigoplus_{i \in \mathcal{I}} M_i$ and $f = (f_{ij}) \in S$ where $f_{ij} \in \text{Hom}_R(M_j, M_i)$. Then $f_{ii}^{-1}(Z_2(M_i))$ is a direct summand of M_i for each $i \in \mathcal{I}$. On the other hand, we have $f^{-1}(Z_2(M)) = \bigoplus_{i \in \mathcal{I}} f_{ii}^{-1}(Z_2(M_i))$. Hence $f^{-1}(Z_2(M))$ is a direct summand of M , as asserted. \square

A ring is called *abelian* if all its idempotents are central. A module is called *abelian* if its endomorphism ring is abelian. It is well known that a module M is abelian if and only if every direct summand of M is fully invariant in M .

Corollary 2.24. *Let $\{M_i\}_{i \in \mathcal{I}}$ be a class of R -modules for an arbitrary index set \mathcal{I} and $\bigoplus_{i \in \mathcal{I}} M_i$ an abelian module. Then $\bigoplus_{i \in \mathcal{I}} M_i$ is Goldie Rickart if and only if M_i is Goldie Rickart for all $i \in \mathcal{I}$.*

Proposition 2.25. *Let M be a Goldie Rickart module with its endomorphism ring von Neumann regular. Then any finite direct sum of copies of M is also Goldie Rickart.*

Proof. Let \mathcal{I} be a finite index set, assume $\mathcal{I} = \{1, 2, \dots, n\}$. By Theorem 2.8, we have $M = Z_2(M) \oplus N$ where N is Rickart. Then $\text{End}_R(N) = eSe$ for some idempotent $e \in S$, and so $\text{End}_R(\bigoplus_{\mathcal{I}} N) = M_n(\text{End}_R(N))$. Since S is von Neumann regular and the von Neumann regularity is Morita invariant, $\text{End}_R(\bigoplus_{\mathcal{I}} N)$ is von Neumann regular. Hence $\bigoplus_{\mathcal{I}} N$ is Rickart by [11, Corollary 3.2]. Also we have $\bigoplus_{\mathcal{I}} M = \bigoplus_{\mathcal{I}} Z_2(M) \oplus (\bigoplus_{\mathcal{I}} N) = Z_2(\bigoplus_{\mathcal{I}} M) \oplus (\bigoplus_{\mathcal{I}} N)$. This implies that $\bigoplus_{\mathcal{I}} M$ is Goldie Rickart. \square

Lemma 2.26. *Let M be a Goldie Rickart module and N a direct summand of M which contains $Z_2(M)$. Then for any direct summand K of M , $N \cap K$ is also a direct summand of M .*

Proof. Let K be any direct summand of M . Then there exist idempotents $e, f \in S$ such that $N = eM$ and $K = fM$. Since $Z_2(M) \subseteq eM$, we have $eM = (1 - e)^{-1}(Z_2(M))$. We claim that $((1 - e)f)^{-1}(Z_2(M)) = (eM \cap fM) \oplus (1 - f)M$. Let $m \in ((1 - e)f)^{-1}(Z_2(M))$. Hence $fm \in (1 - e)^{-1}(Z_2(M))$ and so $m = fm + (1 - f)m \in (eM \cap fM) \oplus (1 - f)M$. This implies that $((1 - e)f)^{-1}(Z_2(M)) \subseteq (eM \cap fM) \oplus (1 - f)M$. For the reverse inclusion, let $x + y \in (eM \cap fM) \oplus (1 - f)M$. Thus $(1 - e)f(x + y) = (1 - e)fx + (1 - e)fy = (1 - e)x + (1 - e)f(1 - f)y \in Z_2(M)$. Then we have $(eM \cap fM) \oplus (1 - f)M \subseteq ((1 - e)f)^{-1}(Z_2(M))$. Since M is Goldie Rickart, $((1 - e)f)^{-1}(Z_2(M))$ is a direct summand of M . Therefore $eM \cap fM$ is a direct summand of M , as required. \square

By virtue of Lemma 2.26, we obtain the next result, and then we give another characterization of Goldie Rickart modules.

Proposition 2.27. *Let M be a Goldie Rickart module. Then M has the summand intersection property for direct summands which contain $Z_2(M)$.*

The converse of Proposition 2.27 does not hold in general, for example the module M in Examples 2.13(1) is Rickart and so it has the summand intersection property by [6, Proposition 2.16], but it is not Goldie Rickart.

Theorem 2.28. *The following are equivalent for a module M .*

- (1) M is Goldie Rickart.
- (2) $t_M(I)$ is a direct summand of M for each finite subset I of S .

Proof. (1) \Rightarrow (2) Let $n \in \mathbb{N}$ and $I = \{f_1, f_2, \dots, f_n\} \subseteq S$. For the proof, we apply induction on n . If $n = 1$, then there is nothing to show. Now let $n > 1$ and suppose the claim holds for $n - 1$. Hence $t_M(J)$ is a direct summand of M where $J = \{f_1, f_2, \dots, f_{n-1}\}$. Clearly, we have $t_M(I) = t_M(J) \cap f_n^{-1}(Z_2(M))$ and $f_n^{-1}(Z_2(M))$ is also a direct summand of M by (1). Since $t_M(J)$ and $f_n^{-1}(Z_2(M))$ contain $Z_2(M)$, by Proposition 2.27, $t_M(I)$ is a direct summand of M .

(2) \Rightarrow (1) Obvious. \square

Proposition 2.29. *Let M be a Goldie Rickart and projective (injective) module. Then for every direct summand N of M , $Z_2(M) + N$ is also a projective (injective) module.*

Proof. Let N be a direct summand of M . By Theorem 2.8, $M = Z_2(M) \oplus K$ for some submodule K of M . Then $Z_2(M) \cap N = Z_2(N)$ is also a direct summand of M due to Lemma 2.26, let $M = Z_2(N) \oplus L$ for some submodule L of M . Hence $Z_2(M) = Z_2(N) \oplus Z_2(L)$ and $N = Z_2(N) \oplus (N \cap L)$. It follows that $Z_2(M) + N = Z_2(N) \oplus Z_2(L) \oplus (N \cap L)$. Since M is projective (injective), $Z_2(N)$, $Z_2(L)$ and $N \cap L$ is also projective (injective), and so $Z_2(M) + N$ is projective (injective). \square

Lemma 2.30. *Let R be a ring, M an R -module and N a submodule of M . Then $(Z(M) + N)/N \subseteq Z(M/N)$ and $(Z_2(M) + N)/N \subseteq Z_2(M/N)$. Moreover, if R is a ring without zero divisors and the submodule N is a torsion R -module, then $(Z(M) + N)/N = Z(M/N)$ and $(Z_2(M) + N)/N = Z_2(M/N)$.*

Proof. It is easy to see that if $x \in Z(M)$, then $x + N \in Z(M/N)$, and if $x \in Z_2(M)$, then $x + N \in Z_2(M/N)$. Hence $(Z(M) + N)/N \subseteq Z(M/N)$ and $(Z_2(M) + N)/N \subseteq Z_2(M/N)$. Now let R be a ring without zero divisors and the submodule N a torsion R -module. Then for any $m \in M$, $r_R(m)$ is essential in $r_R(m + N)$. It follows that $Z(M/N) \subseteq (Z(M) + N)/N$, and so we have $(Z(M) + N)/N = Z(M/N)$. This implies that $Z_2(M/N) \subseteq (Z_2(M) + N)/N$. Thus $(Z_2(M) + N)/N = Z_2(M/N)$. \square

Lemma 2.31. *Let M be a quasi-projective module and N a submodule of M . Then for each $\bar{f} \in \text{End}_R(M/N)$, there exists $f \in S$ such that $(f^{-1}(Z_2(M)) + N)/N \subseteq \bar{f}^{-1}(Z_2(M/N))$. Moreover, if R is a ring without zero divisors and the submodule N is a torsion R -module, then $(f^{-1}(Z_2(M)) + N)/N = \bar{f}^{-1}(Z_2(M/N))$.*

Proof. Let $\bar{f} \in \text{End}_R(M/N)$ and π denote the natural epimorphism from M to M/N . Consider the following diagram

$$\begin{array}{ccc} M & \xrightarrow{\pi} & M/N \\ \downarrow f & & \downarrow \bar{f} \\ M & \xrightarrow{\pi} & M/N \end{array}$$

Since M is quasi-projective, there exists $f \in S$ such that $\bar{f}\pi = \pi f$. Let $m \in f^{-1}(Z_2(M))$. Since $fm \in Z_2(M)$, by Lemma 2.30, $\bar{f}(m + N) = \bar{f}\pi(m) = \pi f(m) = fm + N \in Z_2(M/N)$. Therefore $m + N \in \bar{f}^{-1}(Z_2(M/N))$, and so $(f^{-1}(Z_2(M)) + N)/N \subseteq \bar{f}^{-1}(Z_2(M/N))$. Let R be a ring without zero divisors and the submodule N a torsion R -module. Then for any $m + N \in \bar{f}^{-1}(Z_2(M/N))$, by Lemma 2.30, we have $\bar{f}(m + N) = fm + N \in (Z_2(M) + N)/N$. Hence $m \in f^{-1}(Z_2(M))$, and so $\bar{f}^{-1}(Z_2(M/N)) \subseteq (f^{-1}(Z_2(M)) + N)/N$. \square

Proposition 2.32. *Let R be a ring without zero divisors and M a quasi-projective Goldie Rickart module. If N is a fully invariant submodule of M and a torsion R -module, then M/N is also Goldie Rickart.*

Proof. Let $\bar{f} \in \text{End}_R(M/N)$. By Lemma 2.31, there exists $f \in S$ with $(f^{-1}(Z_2(M)) + N)/N = \bar{f}^{-1}(Z_2(M/N))$. Since M is Goldie Rickart, $f^{-1}(Z_2(M)) = eM$ for some $e^2 = e \in S$. Let π denote the natural epimorphism from M to M/N and consider the following diagram

$$\begin{array}{ccc} M & \xrightarrow{\pi} & M/N \\ \downarrow e & & \downarrow \bar{e} \\ M & \xrightarrow{\pi} & M/N \end{array}$$

Since N is fully invariant, by the Factor Theorem, there exists a unique homomorphism $\bar{e} \in \text{End}_R(M/N)$ such that $\bar{e}\pi = \pi e$. It follows that $\bar{e}^2 = \bar{e}$. Also $\bar{e}(M/N) = (f^{-1}(Z_2(M)) + N)/N$. This completes the proof. \square

3. APPLICATIONS : GOLDIE RICKART RINGS

In this section we study the concept of Goldie Rickart for the ring case. A ring R is called *right Goldie Rickart* if the right R -module R is Goldie Rickart, i.e., for any

$a \in R$, the right ideal $a^{-1}(Z_2(R_R)) = \{b \in R \mid ab \in Z_2(R_R)\}$ is a direct summand of R . As a consequence of Theorem 2.28, a ring R is right Goldie Rickart if and only if $t_R(I)$ is a direct summand of R as a right ideal for each finite subset I of R . Left Goldie Rickart rings are defined similarly. Goldie Rickart rings are not left-right symmetric as the following example shows.

Example 3.1. Consider the ring $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ in [5, (7.22) Example]. It is

shown that R is right nonsingular and $Z(R_R) = \{0, x\}$ where $x = \begin{bmatrix} 0 & \bar{1} \\ 0 & 0 \end{bmatrix}$. Also

$Z({}_R(R/Z(R_R))) = \{0, \overline{m}\}$ where $m = \begin{bmatrix} 0 & 0 \\ 0 & \bar{1} \end{bmatrix} \in R$, and so $Z_2({}_R R) = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$.

Thus $R = Z_2({}_R R) \oplus \begin{bmatrix} \mathbb{Z} & 0 \\ 0 & 0 \end{bmatrix}$. It can be easily shown that $\begin{bmatrix} \mathbb{Z} & 0 \\ 0 & 0 \end{bmatrix}$ is a Rickart left R -module. Therefore R is a left Goldie Rickart ring by Theorem 2.8. On the other hand, for $y = \begin{bmatrix} 2 & \bar{0} \\ 0 & \bar{0} \end{bmatrix} \in R$, $r_R(y) = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ and it is not a direct summand of R as a right ideal. This implies that R is not a right Rickart ring, and so it is not right Goldie Rickart because of right nonsingularity of R .

Remark 3.2. Clearly, every left (right) Rickart ring is left (right) nonsingular, and so it is a left (right) Goldie Rickart ring. But there is a left (right) Goldie Rickart ring which is not left (right) Rickart. For instance, in Example 3.1, the ring R is left Goldie Rickart but not left Rickart. It is obvious that a ring is left (right) Rickart if and only if it is left (right) Goldie Rickart and left (right) nonsingular.

As in the following example, the Goldie Rickart property does not pass on from a module to any its over module in general.

Example 3.3. Consider the ring R in Example 3.1 and let M denote the right R -module R and N the submodule $\begin{bmatrix} 0 & 0 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ of M . Note that $N = \begin{bmatrix} 0 & 0 \\ 0 & \bar{1} \end{bmatrix} M$ for an idempotent $\begin{bmatrix} 0 & 0 \\ 0 & \bar{1} \end{bmatrix} \in S$. Since N is a simple module, obviously it is Rickart. Being N nonsingular, it is also Goldie Rickart. But it is known from Example 3.1, M is not Goldie Rickart.

According to Proposition 2.21, we have the next result.

Proposition 3.4. *Let R be a right Goldie Rickart ring. Then for every idempotent e of R , eR is a Goldie Rickart module.*

Lemma 3.5. *Every finitely generated projective module over a von Neumann regular ring is Goldie Rickart.*

Proof. Let R be a von Neumann regular ring and M a finitely generated projective R -module. Then M is a direct summand of a finitely generated free R -module F . We can see F as $\bigoplus_{i=1}^n R_i$ where $n \in \mathbb{N}$ and $R_i = R$ for all $i = 1, \dots, n$. Since R is von Neumann regular, it is a right Goldie Rickart ring. Hence F is also Goldie Rickart from Proposition 2.25 and so is M due to Proposition 2.21. \square

Proposition 3.6. *Every finitely presented module over a von Neumann regular ring is Goldie Rickart.*

Proof. Let R be a von Neumann regular ring and M a finitely presented R -module. Then M is a flat module. Since M is finitely presented, it is finitely generated and projective. Hence Lemma 3.5 completes the proof. \square

Proposition 3.7. *Every abelian free module over a right Goldie Rickart ring is Goldie Rickart.*

Proof. Let R be a right Goldie Rickart ring and F an abelian free R -module. Assume that F is $\bigoplus_{i \in \mathcal{I}} R_i$ where \mathcal{I} is any index set and $R_i = R$ for all $i \in \mathcal{I}$. Being R Goldie Rickart as an R -module, F is Goldie Rickart from Corollary 2.24. \square

The following theorem gives a characterization of right Goldie Rickart rings in terms of Goldie Rickart modules.

Theorem 3.8. *Let R be a ring and consider the following conditions.*

- (1) *Every R -module is Goldie Rickart.*
- (2) *Every nonsingular R -module is Rickart and $Z_2(R_R)$ is a direct summand of R .*
- (3) *Every projective R -module is Goldie Rickart.*
- (4) *Every free R -module is Goldie Rickart.*
- (5) *R is a right Goldie Rickart ring.*
- (6) *Every cyclic projective R -module is Goldie Rickart.*

Then $(1) \Rightarrow (2) \Rightarrow (3) \Leftrightarrow (4) \Rightarrow (5) \Leftrightarrow (6)$.

Proof. (1) \Rightarrow (2) is clear by Theorem 2.8. (3) \Rightarrow (4) \Rightarrow (5) and (6) \Rightarrow (5) are obvious.

(2) \Rightarrow (4) Let F be a free module. By hypothesis $Z_2(R_R)$ is a direct summand of R , and so $Z_2(F)$ is a direct summand of F . Let $F = Z_2(F) \oplus L$ for some

submodule L of F . Since L is nonsingular, it is Rickart. Then F is Goldie Rickart due to Theorem 2.8.

(4) \Rightarrow (3) Let P be a projective module. Then there exists a free module F and a submodule K of F such that $P \cong F/K$. Hence K is a direct summand of F . Let $F = K \oplus N$ for some submodule N of F . By (4), F is Goldie Rickart and due to Proposition 2.21, N is Goldie Rickart and so is P .

(5) \Rightarrow (6) Let M be a cyclic projective R -module. Then $M \cong I$ for some direct summand right ideal I of R . Since R is right Goldie Rickart, by Proposition 2.21, I is Goldie Rickart and so is M . \square

4. RELATIVELY GOLDIE RICKART MODULES

Example 2.22 shows that a direct sum of Goldie Rickart modules need not be Goldie Rickart. In this section we define relatively Goldie Rickart property in order to investigate when are direct sums of Goldie Rickart modules also Goldie Rickart.

Definition 4.1. *Let M and N be R -modules. M is called N -Goldie Rickart (or relatively Goldie Rickart to N) if for every homomorphism $f : M \rightarrow N$, $f^{-1}(Z_2(N))$ is a direct summand of M .*

Note that in Definition 4.1, $Z_2(M) \leq f^{-1}(Z_2(N))$. It is evident that a module M is Goldie Rickart if and only if it is M -Goldie Rickart.

Theorem 4.2. *Let M and N be R -modules. Then M is N -Goldie Rickart if and only if for any direct summand M_1 of M and any submodule N_1 of N , M_1 is N_1 -Goldie Rickart.*

Proof. Let M_1 be a direct summand of M , N_1 a submodule of N and $f : M_1 \rightarrow N_1$ a homomorphism. Then $M = M_1 \oplus M_2$ for some submodule M_2 of M , and so $f \oplus 0_{|_{M_2}} : M \rightarrow N$, say $g = f \oplus 0_{|_{M_2}}$. Since M is N -Goldie Rickart, $g^{-1}(Z_2(M))$ is a direct summand of M . Now let $m_1 + m_2 \in g^{-1}(Z_2(M))$. It follows that $g(m_1 + m_2) = fm_1 \in Z_2(N) \cap N_1 = Z_2(N_1)$, hence $g^{-1}(Z_2(M)) \subseteq f^{-1}(Z_2(M_1)) \oplus M_2$. Also for any $m_1 + m_2 \in f^{-1}(Z_2(M_1)) \oplus M_2$, we have $g(m_1 + m_2) = fm_1 \in Z_2(N_1) \subseteq Z_2(N)$, and so $f^{-1}(Z_2(M_1)) \oplus M_2 \subseteq g^{-1}(Z_2(M))$. Thus $g^{-1}(Z_2(M)) = f^{-1}(Z_2(M_1)) \oplus M_2$ is a direct summand of M . This implies that $f^{-1}(Z_2(M_1))$ is a direct summand of M_1 . Therefore M_1 is N_1 -Goldie Rickart. The rest is clear. \square

Corollary 4.3. *Let M be a module. Then the following are equivalent.*

- (1) M is Goldie Rickart.

- (2) For any direct summand N of M and any submodule K of M , N is K -Goldie Rickart.
- (3) For any direct summands N and K of M and any $f : M \rightarrow K$, $f|_N^{-1}(Z_2(N))$ is a direct summand of N .

Recall that a module M has C_2 condition if any submodule N of M which is isomorphic to a direct summand of M is also a direct summand. Let M and N be R -modules. M is called N - C_2 (or relatively C_2 to N) if any submodule K of N which is isomorphic to a direct summand of M is a direct summand of N . Hence M has C_2 condition if and only if it is M - C_2 . It is proved in [7, Proposition 2.26] that for modules M and N , M is N - C_2 if and only if for any direct summand K of M and any submodule L of N , K is L - C_2 .

Theorem 4.4. *Let $\{M_i\}_{i \in \mathcal{I}}$ be a class of R -modules where $\mathcal{I} = \{1, 2, \dots, n\}$. Assume that M_i is M_j - C_2 for all $i, j \in \mathcal{I}$. Then $\bigoplus_{i \in \mathcal{I}} M_i$ is a Goldie Rickart module if and only if M_i is M_j -Goldie Rickart for all $i, j \in \mathcal{I}$.*

Proof. The necessity is clear from Theorem 4.2. For the sufficiency, assume that M_i is M_j -Goldie Rickart for all $i, j \in \mathcal{I}$. Without loss of generality we may assume $n = 2$. Let $f = f_{11} + f_{21} + f_{12} + f_{22}$ denote the matrix representation of f where $f_{ij} : M_j \rightarrow M_i$. Then M_1 and M_2 are Goldie Rickart, and so $M_1 = Z_2(M_1) \oplus L_{11}$ and $M_2 = Z_2(M_2) \oplus L_{22}$. Note that L_{11} and L_{22} are Rickart modules. Now for $f_{12} : M_2 \rightarrow M_1$, by assumption, $M_2 = f_{12}^{-1}(Z_2(M_1)) \oplus L_{12}$. Similarly, for $f_{21} : M_1 \rightarrow M_2$, by assumption, $M_1 = f_{21}^{-1}(Z_2(M_2)) \oplus L_{21}$. Since $Z_2(M_1) \leq f_{21}^{-1}(Z_2(M_2))$ and $Z_2(M_2) \leq f_{12}^{-1}(Z_2(M_1))$, we have $f_{21}^{-1}(Z_2(M_2)) = Z_2(M_1) \oplus [f_{21}^{-1}(Z_2(M_2)) \cap L_{11}]$ and $f_{12}^{-1}(Z_2(M_1)) = Z_2(M_2) \oplus [f_{12}^{-1}(Z_2(M_1)) \cap L_{22}]$. Then $M = Z_2(M_1) \oplus Z_2(M_2) \oplus [f_{21}^{-1}(Z_2(M_2)) \cap L_{11}] \oplus [f_{12}^{-1}(Z_2(M_1)) \cap L_{22}] \oplus L_{12} \oplus L_{21}$. Note that $Z_2(M) = Z_2(M_1) \oplus Z_2(M_2)$ and so $[f_{21}^{-1}(Z_2(M_2)) \cap L_{11}] \oplus [f_{12}^{-1}(Z_2(M_1)) \cap L_{22}] \oplus L_{12} \oplus L_{21}$ is nonsingular. $[f_{21}^{-1}(Z_2(M_2)) \cap L_{11}] \oplus L_{21}$ and $[f_{12}^{-1}(Z_2(M_1)) \cap L_{22}] \oplus L_{12}$ are relatively C_2 . Also $[f_{21}^{-1}(Z_2(M_2)) \cap L_{11}] \oplus L_{21}$ and $[f_{12}^{-1}(Z_2(M_1)) \cap L_{22}] \oplus L_{12}$ are relatively Goldie Rickart since they are direct summands of relatively Goldie Rickart modules M_1 and M_2 , respectively. By [7, Theorem 2.29], $[f_{21}^{-1}(Z_2(M_2)) \cap L_{11}] \oplus [f_{12}^{-1}(Z_2(M_1)) \cap L_{22}] \oplus L_{12} \oplus L_{21}$ is Rickart. Therefore, by Theorem 2.8, M is Goldie Rickart. \square

It is well known that every module which its endomorphism ring is von Neumann regular has C_2 condition. In the light of Theorem 4.4, we can weaken the von Neumann regular endomorphism ring condition in Proposition 2.25 as in the following.

Corollary 4.5. *Let M be a Goldie Rickart module with C_2 condition. Then any finite direct sum of copies of M is also Goldie Rickart.*

Proposition 4.6. *Let R be a right Goldie Rickart ring with C_2 condition as an R -module. Then the following hold.*

- (1) *Every finitely generated free R -module is Goldie Rickart.*
- (2) *Every finitely generated projective R -module is Goldie Rickart.*

Proof. (1) Clear from Corollary 4.5.

(2) The condition (1) and Proposition 2.21 complete the proof. \square

Proposition 4.7. *Let $\{M_i\}_{i \in \mathcal{I}}$ be a class of R -modules for an index set \mathcal{I} and N an R -module. Then the following hold.*

- (1) *If N has the summand intersection property for direct summands which contain $Z_2(N)$ and \mathcal{I} is finite, then N is $\bigoplus_{i \in \mathcal{I}} M_i$ -Goldie Rickart if and only if it is M_i -Goldie Rickart for all $i \in \mathcal{I}$.*
- (2) *If N has the strong summand intersection property for direct summands which contain $Z_2(N)$, then N is $\bigoplus_{i \in \mathcal{I}} M_i$ -Goldie Rickart if and only if it is M_i -Goldie Rickart for all $i \in \mathcal{I}$ where \mathcal{I} is arbitrary.*
- (3) *If N has the strong summand intersection property for direct summands which contain $Z_2(N)$, then N is $\prod_{i \in \mathcal{I}} M_i$ -Goldie Rickart if and only if it is M_i -Goldie Rickart for all $i \in \mathcal{I}$ where \mathcal{I} is arbitrary.*

Proof. (1) Let $\mathcal{I} = \{1, 2, \dots, n\}$. The necessity is clear from Theorem 4.2. For the sufficiency, assume that N is M_i -Goldie Rickart for all $i \in \mathcal{I}$ and $f \in \text{Hom}_R(N, \bigoplus_{i \in \mathcal{I}} M_i)$. Let π_i denote the natural projection from $\bigoplus_{i \in \mathcal{I}} M_i$ to M_i for every $i \in \mathcal{I}$. Then $f = (\pi_1 f, \dots, \pi_n f)$. It can be shown that $f^{-1}(Z_2(\bigoplus_{i \in \mathcal{I}} M_i)) = \bigcap_{i \in \mathcal{I}} (\pi_i f)^{-1}(Z_2(M_i))$. By assumption, $(\pi_i f)^{-1}(Z_2(M_i))$ is a direct summand of N for each $i \in \mathcal{I}$. Since $(\pi_i f)^{-1}(Z_2(M_i))$ contains $Z_2(N)$ for each $i \in \mathcal{I}$, by hypothesis, $f^{-1}(Z_2(\bigoplus_{i \in \mathcal{I}} M_i))$ is a direct summand of N , as desired.

(2) and (3) are proved similar to (1). \square

Corollary 4.8. *Let $\{M_i\}_{i \in \mathcal{I}}$ be a class of R -modules where $\mathcal{I} = \{1, 2, \dots, n\}$. Then for every $j \in \mathcal{I}$, M_j is $\bigoplus_{i \in \mathcal{I}} M_i$ -Goldie Rickart if and only if it is M_i -Goldie Rickart for all $i \in \mathcal{I}$.*

Proof. The necessity is clear from Theorem 4.2. For the sufficiency, let $j \in \mathcal{I}$ and M_j be an M_i -Goldie Rickart module for all $i \in \mathcal{I}$. Then M_j is Goldie Rickart, and

so it has the summand intersection property for direct summands which contain $Z_2(M_j)$ by Proposition 2.27. Thus the rest is clear from Proposition 4.7(1). \square

Theorem 4.9. *Let $\{M_i\}_{i \in \mathcal{I}}$ be a class of R -modules where $\mathcal{I} = \{1, 2, \dots, n\}$. Assume that M_i is M_j -injective for all $i < j \in \mathcal{I}$. Then for any R -module N , $\bigoplus_{i \in \mathcal{I}} M_i$ is an N -Goldie Rickart module if and only if M_i is N -Goldie Rickart for all $i \in \mathcal{I}$.*

Proof. Let N be an R -module. The necessity is clear from Theorem 4.2. For the sufficiency, assume M_i is N -Goldie Rickart for all $i \in \mathcal{I}$. Suppose that $n = 2$ and $f \in \text{Hom}_R(M_1 \oplus M_2, N)$. Then $f\iota_{M_i} \in \text{Hom}_R(M_i, N)$ where $\iota_{M_i} : M_i \rightarrow M_1 \oplus M_2$ is the inclusion map, say $f_i = f\iota_{M_i}$ for $i = 1, 2$. Hence $M_1 = f_1^{-1}(Z_2(N)) \oplus L_1$ and $M_2 = f_2^{-1}(Z_2(N)) \oplus L_2$ for some submodules L_1 of M_1 and L_2 of M_2 . Since $f_1^{-1}(Z_2(N)) \oplus f_2^{-1}(Z_2(N)) \subseteq f^{-1}(Z_2(N))$, we have $f^{-1}(Z_2(N)) = f_1^{-1}(Z_2(N)) \oplus f_2^{-1}(Z_2(N)) \oplus [f^{-1}(Z_2(N)) \cap (L_1 \oplus L_2)]$. By Theorem 4.2, L_i is N -Goldie Rickart for $i = 1, 2$. Note that $f_{|L_1 \oplus L_2}^{-1}(Z_2(N)) = f^{-1}(Z_2(N)) \cap (L_1 \oplus L_2)$. Since L_1 is L_2 -injective and $L_1 \cap [f^{-1}(Z_2(N)) \cap (L_1 \oplus L_2)] = 0$, there exists a submodule K of $L_1 \oplus L_2$ such that $L_1 \oplus L_2 = L_1 \oplus K$ and $f^{-1}(Z_2(N)) \cap (L_1 \oplus L_2) \subseteq K$. Clearly, $f_{|K}^{-1}(Z_2(N)) = f^{-1}(Z_2(N)) \cap (L_1 \oplus L_2)$. Also since $K \cong L_2$, K is N -Goldie Rickart. Thus $f_1^{-1}(Z_2(N))$ is a direct summand of K . It follows that $f^{-1}(Z_2(N)) \cap (L_1 \oplus L_2)$ is a direct summand of $L_1 \oplus L_2$. This implies that $f^{-1}(Z_2(N))$ is a direct summand of $M_1 \oplus M_2$. So $M_1 \oplus M_2$ is N -Goldie Rickart. Now suppose that $\bigoplus_{i=1}^{n-1} M_i$ is N -Goldie Rickart and we show $\bigoplus_{i=1}^n M_i$ is also N -Goldie Rickart. Since $\bigoplus_{i=1}^{n-1} M_i$ is M_n -injective and M_n is N -Goldie Rickart, by the preceding discussion, $\bigoplus_{i=1}^n M_i$ is an N -Goldie Rickart module. This completes the proof by the induction on n . \square

We conclude this paper by presenting a result on Goldie Rickart property of $\bigoplus_{i \in \mathcal{I}} M_i$ apart from Theorem 4.4.

Corollary 4.10. *Let $\{M_i\}_{i \in \mathcal{I}}$ be a class of R -modules where $\mathcal{I} = \{1, 2, \dots, n\}$. Assume that M_i is M_j -injective for all $i < j \in \mathcal{I}$. Then $\bigoplus_{i \in \mathcal{I}} M_i$ is a Goldie Rickart module if and only if M_i is M_j -Goldie Rickart for all $i, j \in \mathcal{I}$.*

Proof. The necessity is true by Theorem 4.2. For the sufficiency, assume that M_i is M_j -Goldie Rickart for all $i, j \in \mathcal{I}$. Due to Corollary 4.8, M_i is $\bigoplus_{i \in \mathcal{I}} M_i$ -Goldie Rickart. According to Theorem 4.9, $\bigoplus_{i \in \mathcal{I}} M_i$ is also $\bigoplus_{i \in \mathcal{I}} M_i$ -Goldie Rickart, hence it is Goldie Rickart. \square

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